Chapter 2

Mathematical Background

The principle of relativity states that the mathematical form of the laws of physics remains unchanged (covariant) under general coordinate transformations. The principle is automatically fulfilled if the equations of physics are written as tensor equations. A brief introduction into the concept of tensors will be provided in this chapter. The formalism that will be introduced applies to pseudo-Euclidean Minkowski space time as well as to curved space time and general relativity. Special relativity then translates to invariance of the laws of physics under Lorentz transformations (Lorentz covariance) [1], while general relativity translates to invariance of the laws of physics under general coordinate transformations (general covariance) [2].

2.1 Vectors, one-forms, and tensors

Tensors are multilinear functions of \( q + p \) variables (\( q \) vectors, \( p \) one-forms) \[ T(\vec{V}_1, \ldots, \vec{V}_q; \tilde{W}_1, \ldots, \tilde{W}_p) \] (2.1) where \( \vec{V}_i \in \mathbb{L} \) and \( \tilde{W}_j \in \mathbb{L}^* \) and \( \mathbb{L} \) denotes a linear or vector space. The sum \( r \equiv p + q \) is called the rank of a tensor. The simplest examples of tensor are scalars \( (r = 0) \) and vectors \( (r = 1) \). A linear vector space is defined by the following conditions:

1. Addition: if \( \vec{a} \in \mathbb{L} \) and \( \vec{b} \in \mathbb{L} \), then \( \vec{a} + \vec{b} \in \mathbb{L} \).
2. Multiplication: if \( \vec{a} \in \mathbb{L} \) and \( \lambda \) a real number, then \( \lambda \vec{a} \in \mathbb{L} \).
3. Identity element: there exists an element \( \vec{0} \in \mathbb{L} \) so that \( \vec{a} + \vec{0} = \vec{a} \) for all \( \vec{a} \in \mathbb{L} \).
4. Inverse element: for all \( \vec{a} \in \mathbb{L} \) there exists an element \( -\vec{a} \in \mathbb{L} \) so that \( \vec{a} + (-\vec{a}) = \vec{0} \).
5. Commutativity: \( \vec{a} + \vec{b} = \vec{b} + \vec{a} \).
(6) Distributivity: \( \lambda (\vec{a} + \vec{b}) = \lambda \vec{a} + \lambda \vec{b} \).

The multilinearity of tensors implies that
\[
T(\alpha \vec{a} + \beta \vec{b}) = \alpha T(\vec{a}) + \beta T(\vec{b}).
\] (2.2)

For vectors we introduce the notation
\[
\vec{V} = \sum_\alpha V^\alpha \vec{e}_\alpha \equiv V^\alpha \vec{e}_\alpha ,
\] (2.3)

where, in the second relation on the right hand side, use of Einstein’s summation convention was made. According to this convention, doubly occurring indices are being summed over if one of the indices is positioned as a superscript and the other as a subscript. For instance, no summation is carried out in \( a^\alpha b^\alpha \) or \( a^\alpha b^\beta \) since both summation indices are either subscripts or superscripts. In contrast to this, summations over \( \alpha \) are performed in the following two relations,
\[
a_\alpha b^\alpha = a_1 b^1 + a_2 b^2 + a_3 b^3 ,
\] (2.4)

\[
S^\alpha T_{\alpha \beta} \gamma = S^0 T_{0 \beta} \gamma + S^1 T_{1 \beta} \gamma + S^2 T_{2 \beta} \gamma + S^3 T_{3 \beta} \gamma .
\] (2.5)

The vector components \( V^\alpha \) carry superscripts and are called contravariant, while the basis vectors \( \vec{e}_\alpha \) carry subscripts and are called covariant. One-forms (also referred to as dual vectors) are defined as
\[
\tilde{W} = W_\alpha \tilde{\omega}^\alpha ,
\] (2.6)

where \( W_\alpha \) are the covariant components of the one form and \( \tilde{\omega}^\alpha \) the contravariant basis one-form (dual basis). Instead of writing for vectors
\[
\vec{V} = V_x \vec{e}_x + V_y \vec{e}_y + V_z \vec{e}_z = V_1 \vec{e}_1 + V_2 \vec{e}_2 + V_3 \vec{e}_3 = \sum_{i=1}^3 V_i \vec{e}_i ,
\] (2.7)

we shall be using
\[
\vec{V} = V^x \vec{e}_x + V^y \vec{e}_y + V^z \vec{e}_z = V^1 \vec{e}_1 + V^2 \vec{e}_2 + V^3 \vec{e}_3 \equiv \sum_{i=1}^3 V^i \vec{e}_i
\] (2.8)

throughout this text. Extending this notation to four-dimensional spaces leads to
\[
\vec{V} = V^0 \vec{e}_0 + V^1 \vec{e}_1 + V^2 \vec{e}_2 + V^3 \vec{e}_3 \equiv \sum_{\alpha=0}^3 V^\alpha \vec{e}_\alpha .
\] (2.9)
Substituting \( \vec{V}_i = V_i^\lambda \vec{e}_\lambda \) where \( i = 1, \ldots, q \) and \( \vec{W}_j = W_{jk} \vec{\omega}^k \) where \( j = 1, \ldots, p \) into (2.1) gives

\[
T(\vec{V}_1, \ldots, \vec{V}_q; \vec{W}_1, \ldots, \vec{W}_p) = T(V_1^\lambda \vec{e}_\lambda, \ldots, V_q^\lambda \vec{e}_\lambda; W_{1k_1} \vec{\omega}^{k_1}, \ldots, W_{pk_p} \vec{\omega}^{k_p}) = V_1^\lambda \ldots V_q^\lambda W_{1k_1} \ldots W_{pk_p} T(\vec{e}_{\lambda_1}, \ldots, \vec{e}_{\lambda_q}; \vec{\omega}^{k_1}, \ldots, \vec{\omega}^{k_p}) = 0^\lambda_1 \ldots 0^\lambda_q W_{1k_1} \ldots W_{pk_p} T_{\lambda_1 \ldots \lambda_q}^{k_1 \ldots k_p},
\]

where we have defined

\[
T_{\lambda_1 \ldots \lambda_q}^{k_1 \ldots k_p} \equiv T(\vec{e}_{\lambda_1}, \ldots, \vec{e}_{\lambda_q}; \vec{\omega}^{k_1}, \ldots, \vec{\omega}^{k_p}).
\]

Expressed in the basis spanned by basis vectors and basis one-forms, tensors of rank \( r = p + q \) are given by

\[
T = T_{\lambda_1 \ldots \lambda_q}^{k_1 \ldots k_p} \vec{e}_{\lambda_1} \otimes \ldots \otimes \vec{e}_{\lambda_q} \otimes \vec{\omega}^{k_1} \otimes \ldots \otimes \vec{\omega}^{k_p}.
\]

The set of basis vectors \( \{ \vec{e}_\beta \} \) and the set of basis one-forms \( \{ \vec{\omega}^\alpha \} \) are reciprocal systems of basis vectors. This follows from [3]

\[
\vec{W}(\vec{V}) = \vec{W}(V^\alpha \vec{e}_\alpha) = V^\alpha \vec{W}(\vec{e}_\alpha) \equiv V^\alpha \vec{W}_\alpha
\]

and

\[
\vec{W}(\vec{V}) = W_\alpha \vec{\omega}^\alpha(\vec{V}) = W_\alpha \vec{\omega}^\alpha(V^\beta \vec{e}_\beta) = W_\alpha V^\beta \vec{\omega}^\alpha(\vec{e}_\beta) = W_\alpha V^\alpha.
\]

where \( \vec{W} = W_\alpha \vec{\omega}^\alpha \) has been used. Comparing (2.13) and (2.14) shows that

\[
\vec{\omega}^\alpha(\vec{e}_\beta) = \delta^\alpha_\beta.
\]

A \( (p \ q) \) tensor has \( p \) contravariant and \( q \) covariant indices. A famous example of a \( (1 \ 0) \) tensor is the gradient of a function, \( \Phi(x^\alpha(\tau)) \), with \( \tau \) a parameter,

\[
\frac{d\Phi}{d\tau} = \frac{\partial \Phi}{\partial x^\alpha} \frac{dx^\alpha}{d\tau} = \Phi_{,\alpha} u^\alpha
\]

where \( \Phi_{,\alpha} \) are the components and \( u^\alpha \) the basis of a one-form. Equation (2.16) can also be written as \( d\Phi = \Phi_{,\alpha} dx^\alpha \). To discriminate this one-form from the vector notation, it is customary to write this relation as

\[
d\Phi = \Phi_{,\alpha} \tilde{dx}^\alpha.
\]

A comparison with (2.6) shows that \( \tilde{x}^\alpha = \vec{\omega}^\alpha \). Multiplying (2.17) with \( \vec{e}_\beta \) leads to

\[
d\Phi(\vec{e}_\beta) = \Phi_{,\alpha} \tilde{dx}^\alpha(\vec{e}_\beta),
\]

where \( \tilde{dx}^\alpha(\vec{e}_\beta) \) denotes the action of the one-form on the basis vector.
where \( \tilde{d}\Phi (\tilde{e}_\beta) = (\tilde{d}\Phi)_\beta \) is the component of the one-form and \( \tilde{d}x^\alpha (\tilde{e}_\beta) = \delta^\alpha_\beta \). We therefore have \( (\tilde{d}\Phi)_\beta = \Phi,\beta \). We stress that the components of the one-form \( (\tilde{d}\Phi)_\beta \) transform as
\[
(\tilde{d}\Phi)_\beta = \Phi,\beta = \partial\Phi / \partial x^\beta = \partial x^\gamma \Phi,\gamma = \partial x^\gamma (\tilde{d}\Phi)_\gamma \tag{2.19}
\]
which is the transformation rule defining one-forms. Next we present an example of a \((0,1)\) tensor. By definition, we get from (2.10) for \((0,1)\) tensor
\[
T(\tilde{V}) = T(V^\alpha \tilde{e}_\alpha) = V^\alpha T(\tilde{e}_\alpha) = V^\alpha T_\alpha. \tag{2.20}
\]
Upon replacing \( T \) with \( \tilde{p} \) we obtain
\[
\tilde{p}(\tilde{V}) = V^\alpha \tilde{p}(\tilde{e}_\alpha) = V^\alpha p_\alpha, \tag{2.21}
\]
with \( \tilde{p}(\tilde{e}_\alpha) \equiv p_\alpha \). Writing down this tensor in its respective basis gives \( T = \tilde{p}(\tilde{e}_\alpha) \tilde{\omega}^\alpha \). For three dimensional geometries the metric tensor can be written as
\[
g(\tilde{A}, \tilde{B}) = A^\alpha B^\beta g(\tilde{e}_\alpha, \tilde{e}_\beta) \equiv A^\alpha B^\beta g_{\alpha\beta}, \tag{2.22}
\]
whose basis representation is \( g = g_{\alpha\beta} \tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta \). For three dimensional geometries the metric tensor can be written as
\[
g(d\tilde{x}, d\tilde{x}) = g(dx^i \tilde{e}_i, dx^j \tilde{e}_j) = dx^i dx^j g(\tilde{e}_i, \tilde{e}_j) = g_{ij} dx^i dx^j. \tag{2.23}
\]
The distance between two infinitesimally separated points in a three dimensional geometry is given by the line element
\[
ds^2 \equiv g(d\tilde{x}, d\tilde{x}) = g_{ij} dx^i dx^j = dx^2 + dy^2 + dz^2, \tag{2.24}
\]
so that for a Euclidean geometry the components of \( g_{ij} \) are obtained from \( g_{ij} = \tilde{e}_i \cdot \tilde{e}_j = \delta_{ij} \) as
\[
(g_{ij}) = (\delta_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.25}
\]
The line element of four-dimensional pseudo-Euclidean space time is given by
\[
ds^2 \equiv g(d\tilde{x}, d\tilde{x}) = g_{\mu\nu} dx^\mu dx^\nu = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{2.26}
\]
The components \( g_{\mu\nu} \) of the metric tensor are obtained from \( g_{\mu\nu} = \tilde{e}_\mu \cdot \tilde{e}_\nu = \delta_{\mu\nu} \) so that
\[
(g_{\mu\nu}) = (\delta_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.27}
\]
From the definition of the line element in (2.26) it follows that the metric tensor is symmetric, $g_{\mu\nu} = g_{\nu\mu}$. The contravariant and covariant tensor components of the metric tensor obey

$$g_{\mu\nu} g_{\nu\kappa} = \delta_{\mu}^{\kappa}. \quad (2.28)$$

### 2.2 Raising and lowering indices

For practical reasons, from hereon we replace the symbol $\tilde{\omega}^\nu$ that we used to denote dual basis vectors with $\vec{e}^\nu$. Hence, an arbitrary vector can thus be written in either the coordinate basis system or the dual basis system as

$$\vec{X} = x^\mu \vec{e}_\mu = x_\nu \tilde{e}^\nu. \quad (2.29)$$

Multiplying this relation with $\vec{e}^\kappa$ gives

$$x^\mu \vec{e}_\mu \cdot \vec{e}^\kappa = x_\nu \tilde{e}^\nu \cdot \vec{e}^\kappa. \quad (2.30)$$

Since $\vec{e}_\mu \cdot \vec{e}^\kappa = \delta^\kappa_\mu$ we find from (2.30) that

$$x^\kappa = x_\nu \tilde{e}^\nu \cdot \vec{e}^\kappa = x_\nu g^{\nu\kappa} = g^{\nu\kappa} x_\nu. \quad (2.31)$$

Similarly, upon multiplying $x^\mu \vec{e}_\mu = x_\nu \tilde{e}^\nu$ with $\vec{e}_\kappa$, we obtain

$$x^\mu \vec{e}_\mu \cdot \vec{e}_\kappa = x_\nu \tilde{e}^\nu \cdot \vec{e}_\kappa \quad (2.32)$$

so that

$$x_\kappa = x^\mu \vec{e}_\mu \cdot \vec{e}_\kappa = x^\mu g_{\mu\kappa} = g_{\kappa\mu} x^\mu. \quad (2.33)$$

Using (2.33), we can write $x_\mu \tilde{e}^\mu$ as $x_\mu \tilde{e}^\mu = g_{\mu\nu} x^\nu \vec{e}^\mu = x^\nu \tilde{e}_\nu$ so that

$$\tilde{e}_\nu = g_{\nu\mu} \tilde{e}^\mu. \quad (2.34)$$

Finally, from $x^\mu \vec{e}_\mu = g^{\mu\nu} x_\nu \vec{e}_\mu = x_\nu \vec{e}^\nu$ it follows that

$$\vec{e}^\nu = g^{\nu\mu} \vec{e}_\mu. \quad (2.35)$$

For the pseudo-Euclidean metric of (2.27) it follows that

$$x_0 = -x^0, \quad x_1 = x^1, \quad x_2 = x^2, \quad x_3 = x^3. \quad (2.36)$$

Only the time-like ($x^0$) component of $x^\mu$ changes its sign when switching from contravariant to covariant components while the space-like components $x^i$ ($i = 1, 2, 3$)
remain unchanged. We may thus write for the contravariant and covariant components of \( x^\mu \)
\[
(x^\mu) = (x^0, x^1, x^2, x^3) = (t, x, y, z),
\]
(2.37)
\[
(x_\mu) = (-x^0, x^1, x^2, x^3) = (-t, x, y, z),
\]
(2.38)
respectively. Setting \( a = g \) in \( a_{\nu\kappa} g^{\nu\kappa} = a^\mu_\kappa \) we obtain the important relation
\[
g^{\mu\nu} g_{\nu\kappa} = g^\mu_\kappa,
\]
(2.39)
from which it follows that
\[
g^\mu_\nu = \delta^\mu_\nu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases}.
\]
(2.40)
We also note that
\[
\delta^{\mu \nu} = g^{\mu \kappa} \delta^\kappa_\nu = g^{\mu \nu}
\]
(2.41)
so that \( \delta^{00} = g^{00} = -1 \) and \( \delta^{ii} = g^{ii} = +1 \). The line element (2.26) can thus be written as
\[
ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu = dx_\nu \, dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2.
\]
(2.42)

### 2.3 Coordinate Transformations

Given are two coordinate systems \( S \) and \( S' \) with coordinates \( \{x^\nu\} \) and \( \{x'^\nu\} \), respectively. The coordinate differentials of both coordinate systems are thus related by
\[
dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} \, dx^\nu \equiv \Lambda^\mu_\nu \, dx^\nu.
\]
(2.43)
For the inverse transformation we get
\[
dx^\nu = \frac{\partial x^\nu}{\partial x'^\mu} \, dx'^\mu \equiv \Lambda^\nu_\mu \, dx'^\mu.
\]
(2.44)
Writing down a vector in these primed (new) as well as unprimed (old) coordinates leads to
\[
dx^\mu \, \vec{e}'_\alpha = dx^\beta \, \vec{e}_\beta,
\]
(2.45)
from which it follows that
\[
\Lambda^\alpha_\beta \, dx^\beta \, \vec{e}'_\alpha = dx^\beta \, \vec{e}_\beta \quad \Rightarrow \quad (\Lambda^\alpha_\beta \, \vec{e}'_\alpha - \vec{e}_\beta) \, dx^\beta = 0.
\]
(2.46)
2.3. Coordinate transformations

From this relation we read off that the covariant coordinate basis vectors transform as
\[ \vec{e}_\beta = \Lambda^\alpha_\beta \vec{e}'_\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} \vec{e}'_\alpha. \]  
(2.47)

From (2.45) follows also that
\[ dx'^\alpha \vec{e}'_\alpha = \Lambda^\alpha_\beta dx^\alpha \vec{e}_\beta, \]  
(2.48)
which leads to
\[ (\vec{e}'_\alpha - \Lambda^\alpha_\beta \vec{e}_\beta) \, dx'^\alpha = 0. \]  
(2.49)

The primed covariant basis vectors therefore transform as
\[ \vec{e}'_\alpha = \Lambda^\alpha_\beta \vec{e}_\beta = \frac{\partial x^\beta}{\partial x'^\alpha} \vec{e}_\beta. \]  
(2.50)

In the following we illustrate the transformation laws derived in (2.47) and (2.50) for two well-know systems of coordinates. These are Cartesian coordinates, in which case \( \{x^\beta\} \equiv \{x_1, x_2\} \equiv \{x, y\} \), and plane polar coordinates for which \( \{x'^\alpha\} \equiv \{x_1, x_2\} \equiv \{r, \theta\} \) with \( x_1 = r \cos \theta \) and \( x_2 = r \sin \theta \). For \( \alpha = 1 \) we obtain from (2.50) for the basis vectors in polar coordinates
\[ \vec{e}'_1 = \frac{\partial x^1}{\partial x'^1} \vec{e}_1 + \frac{\partial x^2}{\partial x'^1} \vec{e}_2 = \frac{\partial x}{\partial r} \vec{e}_x + \frac{\partial y}{\partial r} \vec{e}_y, \]  
(2.51)
and after carrying out the partial derivatives in (2.51), we obtain
\[ \vec{e}_r = \cos \theta \vec{e}_x + \sin \theta \vec{e}_y. \]  
(2.52)

Similarly, for \( \alpha = 2 \) we obtain from (2.50)
\[ \vec{e}'_2 = \frac{\partial x^1}{\partial x'^2} \vec{e}_1 + \frac{\partial x^2}{\partial x'^2} \vec{e}_2 = \frac{\partial x}{\partial \theta} \vec{e}_x + \frac{\partial y}{\partial \theta} \vec{e}_y, \]  
(2.53)
which leads to
\[ \vec{e}_\theta = -r \sin \theta \vec{e}_x + r \cos \theta \vec{e}_y. \]  
(2.54)

In the next step, we want to find out how the components of vectors (rather than the basis vectors) transform. For this purpose recall that \( V'^\alpha \vec{e}'_\alpha = V^\beta \vec{e}_\beta \). Making use of this relation together with (2.47), it follows that
\[ V'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} V^\beta, \]  
(2.55)
and by means of (2.50) one thus obtains

\[ V^\beta = \frac{\partial x^\beta}{\partial x'^\alpha} V'^\alpha. \] (2.56)

A comparison of equations (2.47) and (2.50) with equations (2.55) and (2.56) shows that the components of vectors transform in the opposite manner to that of basis vectors. That is why ordinary vectors are called contravariant.

We finish this section by mentioning that

\[ x'^\alpha \bar{e}'^\alpha = (\Lambda^\alpha_\beta x^\beta) (\Lambda_\alpha^\gamma \bar{e}_\gamma) = x^\gamma \bar{e}_\gamma \] (2.57)

implies for the transformation matrices

\[ \Lambda^\alpha_\beta \Lambda_\alpha^\gamma = \delta^\gamma_\gamma, \] (2.58)

or, expressed in terms of partial derivatives,

\[ \frac{\partial x'^\alpha}{\partial x^\beta} \frac{\partial x^\gamma}{\partial x'^\alpha} = \delta^\beta_\gamma. \] (2.59)

### 2.4 Transformation properties of one-forms

As discussed in section 2.1, one-forms consist of covariant vector components and contravariant basis vector. We can thus infer from the transformation laws derived in (2.50) and (2.55) that the components and basis vectors of one-forms transform as

\[ x'^\alpha \tilde{e}'^\alpha = (\Lambda^\alpha_\beta x^\beta) (\Lambda_\alpha^\gamma \tilde{e}_\gamma) = x^\gamma \tilde{e}_\gamma \] (2.57)

implies for the transformation matrices

\[ \Lambda^\alpha_\beta \Lambda_\alpha^\gamma = \delta^\gamma_\gamma, \] (2.58)

or, expressed in terms of partial derivatives,

\[ \frac{\partial x'^\alpha}{\partial x^\beta} \frac{\partial x^\gamma}{\partial x'^\alpha} = \delta^\beta_\gamma. \] (2.59)

### 2.5 Transformation properties of tensors

Tensors are defined as linear combinations of \( p \) one-forms and \( q \) vectors, as defined in (2.10).

\[ T'(\tilde{V}'_1, \ldots, \tilde{V}'_q; \tilde{W}'_1, \ldots, \tilde{W}'_p) = T^d(V^{d}_1 \tilde{e}'_{1}^d, \ldots, V^{d}_q \tilde{e}'_{q}^d; W^{d}_1 \bar{e}'_{1}^d, \ldots, W^{d}_p \bar{e}'_{p}^d). \] (2.62)

Because of the multilinearity of tensors, the right-hand-side of (2.62) can be written as

\[ V^{d}_1 \ldots V^{d}_q W^{d}_1 \ldots W^{d}_p T'(\tilde{e}'_{1}^d, \ldots, \tilde{e}'_{q}^d, \bar{e}'_{1}^d, \ldots, \bar{e}'_{p}^d). \] (2.63)
By means of the definition

\[ T'_{\lambda_1' \ldots \lambda_q'} \equiv T^{\kappa_1' \ldots \kappa_p'}_{\lambda_1' \ldots \lambda_q'} = T'(\vec{e}'_{\lambda_1'}, \ldots, \vec{e}'_{\lambda_q'}, \vec{e}_1', \ldots, \vec{e}_p') \]  

and equation (2.50), it follows that the components of tensors transform as

\[ T'_{\lambda_1' \ldots \lambda_q'} = \frac{\partial x^{\beta_1}}{\partial x'^{\lambda_1'}} \cdots \frac{\partial x^{\beta_q}}{\partial x'^{\lambda_q'}} \frac{\partial x^{\kappa_1'}}{\partial x'^{\tau_1'}} \cdots \frac{\partial x^{\kappa_p'}}{\partial x'^{\tau_p'}} T_{\tau_1' \ldots \tau_p'}. \]  

Examples of famous tensors are the Einstein tensor, \( G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \), and the energy-momentum tensor, \( T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu - g^{\mu\nu} P \), which will be derived in chapter 7. The covariance of Maxwell’s equations can be verified rather easily if the Maxwell equations are written as tensor equations, which is our next goal. For this purpose we set the vacuum permeability and vacuum permittivity \( \mu_0 = \varepsilon_0 = 1 \). We also set the speed of light \( c = 1 \). Gauss’ law for the electric field is then given by

\[ \vec{\nabla} \cdot \vec{E} = 4\pi \rho, \]  

Gauss’ law for the magnetic field reads

\[ \vec{\nabla} \cdot \vec{B} = 0, \]  

for Ampere’s law one has

\[ \vec{\nabla} \times \vec{B} = 4\pi \vec{j} + \frac{\partial \vec{E}}{\partial t}, \]  

and Faraday’s law is given by

\[ \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \]  

The equation of continuity, which describes the conservation of electric charge, has the form

\[ \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0. \]  

Maxwell’s equations (2.66) to (2.69) can be written in a more compact and elegant form, which is useful when demonstrating their Lorentz covariance, by introducing the anti-symmetric \((2,0)\) tensors \( F \) and \( \tilde{F} \) on space time coordinates by

\[ \begin{pmatrix} (F^{\mu\nu}) \\ (\tilde{F}^{\mu\nu}) \end{pmatrix} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}, \]  

(2.71)
\[
\left(\tilde{F}^{\mu \nu}\right) = \begin{pmatrix}
0 & -B^3 & -B^2 & -B^1 \\
B^1 & 0 & E^3 & -E^2 \\
B^2 & -E^3 & 0 & E^1 \\
B^3 & E^2 & -E^1 & 0
\end{pmatrix}.
\] (2.72)

With the help of these tensors, Maxwell’s equations (2.66) to (2.69) can be compactly written as
\[
\partial_\mu F^{\mu \nu} = 4\pi j^\nu, \quad \partial_\mu \tilde{F}^{\mu \nu} = 0,
\] (2.73)

where the four-dimensional electric current density \(j^\mu\) is defined by
\[
j^\mu = (\rho, \vec{j}) \equiv (\rho, j^1, j^2, j^3),
\] (2.74)

and the components of the electric and magnetic fields are given by \(\vec{E} = (E^1, E^2, E^3)\) and \(\vec{B} = (B^1, B^2, B^3)\), respectively. The equation of continuity (2.70) then reads \(j_{\mu \mu} = 0\). An even more compact notation of Maxwell’s equations, which makes only use of \(F^{\mu \nu}\) defined in (2.71), is given by
\[
F^{\mu \nu, \mu} = 4\pi j^\nu, \quad F_{\mu \nu, \lambda} + F_{\nu \lambda, \mu} + F_{\lambda \mu, \nu} = 0.
\] (2.75)

The derivatives in the equations above are four-derivatives defined as
\[
(\partial_\mu) \equiv \left(\frac{\partial}{\partial x^\mu}\right) = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)
\] (2.76)

Since \(\partial_\mu = g^{\mu \nu} \partial_\nu\), the contravariant components of the four-derivative in pseudo-Euclidean space time are given by
\[
(\partial^\mu) \equiv \left(\frac{\partial}{\partial x_\mu}\right) = \left(-\frac{\partial}{\partial t}, \vec{\nabla}\right).
\] (2.77)

Combining (2.76) with (2.77) leads to
\[
\partial_\mu \partial_\nu = g_{\mu \nu} \partial_\nu \partial_\mu = -\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2 \equiv \Box^2.
\] (2.78)

Define 4-vector potential \(A^\mu\) as \((A^\mu) = (\phi, \vec{A})\), where \(\phi\) denotes the electrostatic potential and \(\vec{A}\) the vector potential. The electric \((E^i = -\delta^{ij} \partial_j \phi - \partial_0 A^i)\) and magnetic fields are then given by
\[
\vec{B} = \vec{\nabla} \times \vec{A}, \quad \vec{E} = -\vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t}
\] (2.79)

Define the covariant electromagnetic field tensor as
\[
F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.
\] (2.80)
Its components are given by

\[
(F^\mu_\nu) = \begin{pmatrix}
F_{00} & F_{01} & F_{02} & F_{03} \\
F_{10} & F_{11} & F_{12} & F_{13} \\
F_{20} & F_{21} & F_{22} & F_{23} \\
F_{30} & F_{31} & F_{32} & F_{33}
\end{pmatrix} = \begin{pmatrix}
0 & -E^1 & -E^2 & -E^3 \\
-E^1 & 0 & B^3 & -B^2 \\
-E^2 & B^3 & 0 & B^1 \\
E^3 & B^2 & -B^1 & 0
\end{pmatrix}.
\]

The covariant components of the Maxwell tensor follow from \(F^\kappa_\lambda = g^{\kappa_\mu} g^{\lambda_\nu} F_{\mu_\nu}\) as

\[
(F^\mu_\nu) = \begin{pmatrix}
0 & -E^1 & -E^2 & -E^3 \\
-E^1 & 0 & B^3 & -B^2 \\
-E^2 & B^3 & 0 & B^1 \\
E^3 & B^2 & -B^1 & 0
\end{pmatrix}.
\]

The components of electric and magnetic fields, \(\vec{E}'^\mu\) and \(\vec{B}'^\mu\), in some other Lorentz reference frame are obtained from

\[
A'^\mu = \Lambda^\mu_\nu A^\nu, \quad F'^{\mu_\nu} = \Lambda^\mu_\sigma \Lambda^{\nu_\rho} F^{'\sigma_\rho},
\]

Examples of tensors are the Levi-Civita symbol, which is a tensor of type \((0,4)\). It is defined as

\[
\epsilon^{\alpha_\beta_\gamma_\delta} = \begin{cases} 
1 & \text{if } \alpha \beta \gamma \delta \text{ is an even permutation of } 0123 \\
-1 & \text{if } \alpha \beta \gamma \delta \text{ is an odd permutation of } 0123 \\
0 & \text{otherwise (} \alpha = \beta, \ \text{or } \beta = \gamma, \ \text{or } \gamma = \delta \end{cases}.
\]

Being a \((0,4)\) object, the Levi-Civita tensor transforms as

\[
\epsilon^{\alpha_\beta_\gamma_\delta} = \frac{\partial x^\mu}{\partial x'^{\alpha}} \frac{\partial x^\nu}{\partial x'^{\beta}} \frac{\partial x^\kappa}{\partial x'^{\gamma}} \frac{\partial x^\lambda}{\partial x'^{\delta}} \epsilon_{\mu_\nu_\kappa_\lambda}.
\]

The totally contravariant Levi-Civita tensor is obtained from (2.83) as

\[
\epsilon_{\alpha_\beta_\gamma_\delta} = g^{\alpha_\mu} g^{\beta_\nu} g^{\gamma_\kappa} g^{\delta_\lambda} \epsilon_{\mu_\nu_\kappa_\lambda}.
\]

From (2.83) and (2.84) one finds that \(\epsilon_{\alpha_\beta_\gamma_\delta} \epsilon^{\alpha_\beta_\gamma_\delta} = -24\) and that \(\epsilon_{\alpha_\beta_\gamma_\delta} \epsilon^{\alpha_\beta_\gamma_\epsilon} = -6 \delta^\epsilon_\delta\).

Another example of a tensor is the widely used Kronecker delta. It is a tensor of type \((1,1)\), since it transforms as

\[
\delta^\mu_\nu = \frac{\partial x'^\mu}{\partial x^\kappa} \frac{\partial x^\lambda}{\partial x'^\nu} \delta^\kappa_\lambda.
\]

The last example of a tensor concerns the metric tensor. Invariance of \(ds^2\) under (general) coordinate transformations means that \(dx'_\mu \, dx'^\mu = dx'_\mu \, dx'^\mu\) and hence

\[
g_{\mu_\nu} = \frac{\partial x'^\kappa}{\partial x^\mu} \frac{\partial x^\lambda}{\partial x'^\nu} g_{k_\lambda},
\]

which is nothing else but the transformation rule of a totally covariant tensor of rank 2.